Diffeomorphism-invariant Covariant Hamiltonians of a pseudo-Riemannian Metric and a Linear Connection

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Abstract

Let $\boldsymbol{M} \to \boldsymbol{N}$ (resp. $\boldsymbol{C} \to \boldsymbol{N}$) be the fibre bundle of pseudo-Riemannian metrics of a given signature (resp. the bundle of linear connections) on an orientable connected manifold $\boldsymbol{N}$. A geometrically defined class of first-order Ehresmann connections on the product fibre bundle $\boldsymbol{M} \times_{\boldsymbol{N}} \boldsymbol{C}$ is determined such that, for every connection $\boldsymbol{\gamma}$ belonging to this class and every $\mathbf{Diff}\,\boldsymbol{N}$-invariant Lagrangian density $\boldsymbol{\Lambda}$ on $\boldsymbol{J^1(M \times_N C)}$, the corresponding covariant Hamiltonian $\boldsymbol{\Lambda}^{\gamma}$ is also $\mathbf{Diff}\,\boldsymbol{N}$-invariant. The case of $\mathbf{Diff}\,\boldsymbol{N}$-invariant second-order Lagrangian densities on $\boldsymbol{J^2M}$ is also studied and the results obtained are then applied to Palatini and Einstein-Hilbert Lagrangians.

PACS codes: 02.30.Xx, 02.40.Hw, 02.40.Ma, 02.40.Vh, 04.20.Fy, 04.50.+h

Mathematics Subject Classification 2010: Primary: 58E30; Secondary: 58A20, 58J70, 83C05

Key words and phrases: Covariant Hamiltonian density, Hamilton-Cartan formalism, Diffeomorphism invariance, Jet bundles, Lagrangian density, Poincaré-Cartan form.

Acknowledgements: Supported by Ministerio de Ciencia e Innovación of Spain, under grant #MTM2008–01386.

1 Introduction

In Mechanics, the Hamiltonian function attached to a Lagrangian density $\Lambda = L(t, q^i, \dot{q}^i)dt$ on $\mathbb{R} \times TQ$ is given by $H = \dot{q}^i \partial L / \partial \dot{q}^i - L$, but—as it was early observed in [**16**]—this is not an invariant definition if an arbitrary fibred manifold $t\colon E \to \mathbb{R}$ is considered (thus generalizing the notion of an absolute time) instead of the direct product bundle $\mathbb{R} \times Q \to \mathbb{R}$; e.g., see [**7**], [**23**], [**25**] for this point of view. In this case, an Ehresmann connection is needed in order to lift the vector field $\partial/\partial t$ from $\mathbb{R}$ to E, and the Hamiltonian is then defined by applying the Poincaré-Cartan form attached to Λ to the horizontal lift of $\partial/\partial t$.

In the field theory—where no distinguished vector field exists on the base manifold—the need of an Ehresmann connection is even greater, in order to attach a covariant Hamiltonian to each Lagrangian density; e.g., see [**24**, 4.1], [**23**], and the definitions below.

Let $p\colon E \to N$ be an arbitrary fibred manifold over a connected manifold N, $n = \dim N$, $\dim E = m+n$, oriented by $v_n = dx^1 \wedge \cdots \wedge dx^n$. Throughout this paper, Latin (resp. Greek) indices run from 1 to n (resp. m). An Ehresmann connection on a fibred manifold $p\colon E \to N$ is a differential 1-form γ on E taking values in the vertical sub-bundle $V(p)$ such that $\gamma(X) = X$ for every $X \in V(p)$ (e.g., see [**23**], [**24**], [**32**], [**34**]). Once an Ehresmann connection γ is given, a decomposition of vector bundles holds $T(E) = V(p) \oplus \ker\gamma$, where $\ker\gamma$ is called the horizontal sub-bundle determined by γ. In a fibred coordinate system (x^j, y^α) for p, an Ehresmann connection can be written as

$$\gamma = (dy^\alpha + \gamma_j^\alpha dx^j) \otimes \frac{\partial}{\partial y^\alpha}, \quad \gamma_j^\alpha \in C^\infty(E).$$

According to [**24**], the covariant Hamiltonian Λ^γ associated to a Lagrangian density on J^1E, $\Lambda = Lv_n$, $L \in C^\infty(J^1E)$, with respect to γ is the Lagrangian density defined by,

$$\Lambda^\gamma = \left((p_0^1)^*\gamma - \theta\right) \wedge \omega_\Lambda - \Lambda, \tag{1}$$

where, $p_0^1\colon J^1E \to J^0E = E$ is the projection mapping, $\theta = \theta^\alpha \otimes \partial/\partial y^\alpha$, $\theta^\alpha = dy^\alpha - y_i^\alpha dx^i$ is the $V(p)$-valued 1-form on J^1E associated with the contact structure, written on a fibred coordinate system (x^i, y^α), and ω_Λ is the Legendre form attached to Λ, i.e., the $V^*(p)$-valued p^1-horizontal $(n-1)$-form on J^1E given by

$$\omega_\Lambda = (-1)^{i-1}\frac{\partial L}{\partial y_i^\alpha} i_{\partial/\partial x^i} v_n \otimes dy^\alpha,$$

where $(x^i, y^\alpha; y_i^\alpha)$ is the coordinate system induced from (x^i, y^α) on the 1-jet bundle and $p^1\colon J^1E \to N$ is the projection on the base manifold. Locally,

$$\Lambda^\gamma = \left((\gamma_i^\alpha + y_i^\alpha)\frac{\partial L}{\partial y_i^\alpha} - L\right) dx^1 \wedge \cdots \wedge dx^n. \tag{2}$$

From (1) we obtain the following decomposition of the Poincaré-Cartan form attached to Λ (e.g., see [**17**], [**23**], [**27**]): $\Theta_\Lambda = \theta \wedge \omega_\Lambda + \Lambda = (p_0^1)^*\gamma \wedge \omega_\Lambda - \Lambda^\gamma$.

A diffeomorphism $\Phi\colon E \to E$ is said to be an automorphism of p if there exists $\phi \in \operatorname{Diff} N$ such that $p \circ \Phi = \phi \circ p$. The set of such automorphisms is denoted by $\operatorname{Aut}(p)$ and its Lie algebra is identified to the space $\operatorname{aut}(p) \subset \mathfrak{X}(E)$ of p-projectable vector fields on E. Given a subgroup $\mathcal{G} \subseteq \operatorname{Aut}(p)$, a Lagrangian density Λ is said to be $\mathcal{G}$-invariant if $(\Phi^{(1)})^*\Lambda = \Lambda$ for every $\Phi \in \mathcal{G}$, where $\Phi^{(1)}\colon J^1E \to J^1E$ denotes the 1-jet prolongation of Φ. Infinitesimally, the $\mathcal{G}$-invariance equation can be reformulated as $L_{X^{(1)}}\Lambda = 0$ for every $X \in \operatorname{Lie}(\mathcal{G})$, $X^{(1)}$ denoting the 1-jet prolongation of the vector field X.

When a group $\mathcal{G}$ of transformations of E is given, a natural question arises:

- Determine a class—as small as possible— of Ehresmann connections γ such that Λ^γ is $\mathcal{G}$-invariant for every $\mathcal{G}$-invariant Lagrangian density Λ.

Below we tackle this question in the framework of General Relativity, i.e., the group $\mathcal{G}$ is the group of all diffeomorphisms of the ground manifold N acting in a natural way either on the bundle of pseudo-Riemannian metrics $p_M\colon M = M(N) \to N$ of a given signature (n^+, n^-), $n^+ + n^- = n$, or on the product bundle $p\colon M \times_N C \to N$, where $p_C\colon C = C(N) \to N$ is the bundle of linear connections on N. Namely, we solve the following two problems:

(P): Determine a class—as small as possible— of Ehresmann connections γ such that for every $\mathrm{Diff}N$-invariant first-order Lagrangian density Λ on the bundle $J^1(M\times_N C)$, the corresponding covariant Hamiltonian Λ^γ is also $\mathrm{Diff}N$-invariant.

Similarly to the problem **(P)**, we formulate the corresponding problem on J^2M as follows:

(P2): Determine a class of second-order Ehresmann connections γ^2 on M such that for every $\mathrm{Diff}N$-invariant second-order Lagrangian density Λ on the bundle J^2M, the corresponding covariant Hamiltonian Λ^{γ^2}—defined in (42)—is also $\mathrm{Diff}N$-invariant.

Essentially, a class of first-order Ehresmann connections on the bundle $M\times_N C$ is obtained, defined by the conditions (C_M) and (C_C) below (see Propositions 3.4 and 3.5), solving the problem **(P)**. This class of connections also helps to solve **(P2)** by means of a natural isomorphism between J^1M and $M\times_N C^{\mathrm{sym}}$, where C^{sym} denotes the sub-bundle of symmetric connections on N (cf. Theorem 4.1). Finally, this approach is applied to Palatini and Einstein-Hilbert Lagrangians ([**3**], [**4**]), obtaining results compatible with their usual Hamiltonian formalisms.

2 Invariance under diffeomorphisms

2.1 Preliminaries

2.1.1 Jet-bundle notations

Let $p^k\colon J^kE\to N$ be the k-jet bundle of local sections of an arbitrary fibred manifold $p\colon E\to N$, with projections $p^k_l\colon J^kE\to J^lE$, $p^k_l(j^k_xs)=j^l_xs$, for $k\geq l$, j^k_xs denoting the k-jet at x of a section s of p defined on a neighbourhood of $x\in N$.

A fibred coordinate system (x^i,y^α) on V induces a coordinate system (x^i,y^α_I), $I=(i_1,\dots,i_N)\in\mathbb{N}^n$, $0\leq|I|=i_1+\cdots+i_N\leq r$, on $(p^r_0)^{-1}(V)=J^rV$ as follows: $y^\alpha_I(j^r_xs)=(\partial^{|I|}(y^\alpha\circ s)/\partial x^I)(x)$, with $y^\alpha_0=y^\alpha$.

Every morphism $\Phi\colon E\to E'$ whose associated map $\phi\colon N\to N'$ is a diffeomorphism, induces a map

$$\begin{aligned}&\Phi^{(r)}\colon J^rE\to J^rE',\\&\Phi^{(r)}(j^r_xs)=j^r_{\phi(x)}(\Phi\circ s\circ\phi^{-1}).\end{aligned}\tag{3}$$

If Φ_t is the flow of a vector field $X \in \mathrm{aut}(p)$, then $\Phi_t^{(r)}$ is the flow of a vector field $X^{(r)} \in \mathfrak{X}(J^r E)$, called the infinitesimal contact transformation of order r associated to the vector field X. The mapping

$$\mathrm{aut}(p) \ni X \mapsto X^{(r)} \in \mathfrak{X}(J^r E)$$

is an injection of Lie algebras, namely, one has

$$\begin{aligned}
&(\lambda X + \mu Y)^{(r)} = \lambda X^{(r)} + \mu Y^{(r)}, \\
&[X, Y]^{(r)} = [X^{(r)}, Y^{(r)}], \\
&\forall \lambda, \mu \in \mathbb{R},\ \forall X, Y \in \mathrm{aut}(p).
\end{aligned}$$

In particular, for $r = 1$,

$$X = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}, \quad u^i \in C^\infty(N), v^\alpha \in C^\infty(E),$$

$$X^{(1)} = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha}, \quad v_i^\alpha = \frac{\partial v^\alpha}{\partial x^i} + y_i^\beta \frac{\partial v^\alpha}{\partial y^\beta} - y_k^\alpha \frac{\partial u^k}{\partial x^i}.$$

2.1.2 Coordinates on $M(N)$, $F(N)$, $C(N)$

Every coordinate system (x^i) on an open domain $U \subseteq N$ induces the following coordinate systems:

1) (x^i, y_{jk}) on $(p_M)^{-1}(U)$, where $p_M \colon M \to N$ is the bundle of metrics of a given signature, and the functions $y_{jk} = y_{kj}$ are defined by,

(4) $$g_x = \sum_{i \le j} y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x,\ \forall g_x \in (p_M)^{-1}(U).$$

2) (x^i, x_j^i) on $(p_F)^{-1}(U)$, where $p_F \colon F(N) \to N$ is the bundle of linear frames on N, and the functions x_j^i are defined by,

$$u = \left((\partial/\partial x^1)_x, \ldots, (\partial/\partial x^n)_x\right) \cdot \left(x_j^i(u)\right),\ x = p_F(u), \forall u \in (p_F)^{-1}(U),$$

or equivalently,

(5) $$u = (X_1, \ldots, X_N) \in F_x(N),\ X_j = x_j^i(u) \left(\frac{\partial}{\partial x^i}\right)_x, \quad 1 \le j \le n.$$

3) (x^i, A_{kl}^j) on $(p_C)^{-1}(U)$, where $p_C \colon C \to N$ is the bundle of linear connections on N, and the functions A_{kl}^j are defined as follows. We first recall some basic facts. Connections on $F(N)$ (i.e., linear connections of N) are the splittings of the Atiyah sequence (cf. [**2**]),

$$0 \to \mathrm{ad}F(N) \to T_{Gl(n,\mathbb{R})}F(N) \xrightarrow{(p_F)_*} TN \to 0,$$

where

a) $\mathrm{ad}F(N) = T^*N \otimes TN$ is the adjoint bundle,

b) $T_{Gl(n,\mathbb{R})}(F(N)) = T(F(N))/Gl(n,\mathbb{R})$, and

c) $\mathrm{gau}F(N) = \Gamma(N, \mathrm{ad}F(N))$ is the gauge algebra of $F(N)$.

We think of $\mathrm{gau}F(N)$ as the 'Lie algebra' of the gauge group $\mathrm{Gau}F(N)$. Moreover, $p_C \colon C \to N$ is an affine bundle modelled over the vector bundle $\otimes^2 T^*N \otimes TN$. The section of p_C induced tautologically by the linear connection Γ is denoted by $s_\Gamma \colon N \to C$. Every $B \in \mathfrak{gl}(n,\mathbb{R})$ defines a one-parameter group $\varphi_t^B \colon U \times Gl(n,\mathbb{R}) \to U \times Gl(n,\mathbb{R})$ of gauge transformations by setting (cf. [**5**]), $\varphi_t^B(x,\Lambda) = (x, \exp(tB)\cdot\Lambda)$. Let us denote by $\bar{B} \in \mathrm{gau}(p_F)^{-1}(U)$ the corresponding infinitesimal generator. If (E_j^i) is the standard basis of $\mathfrak{gl}(n,\mathbb{R})$, then $\bar{E}_j^i = \sum_{h=1}^n x_h^j \partial/\partial x_h^i$, for $i,j = 1,\dots,n$, is a basis of $\mathrm{gau}(p_F)^{-1}(U)$. Let $\tilde{E}_j^i = \bar{E}_j^i \bmod G$ be the class of $\bar{E}_j^i$ on $\mathrm{ad}F(N)$. Unique smooth functions A_{jk}^i on $(p_C)^{-1}(U)$ exist such that,

$$
\begin{aligned}
s_\Gamma\left(\frac{\partial}{\partial x^j}\right) &= \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma)\tilde{E}_k^i \\
&= \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma) x_h^k \frac{\partial}{\partial x_h^i},
\end{aligned}
\tag{6}
$$

for every s_Γ and $A_{jk}^i(\Gamma_x) = \Gamma_{jk}^i(x)$, where Γ_{jk}^i are the Christoffel symbols of the linear connection Γ in the coordinate system (x^i), see [**20**, III, Poposition 7.4].

2.2 Natural lifts

Let $f_M \colon M \to M$, cf. [**30**] (resp. $\tilde{f} \colon F(N) \to F(N)$, cf. [**20**, p. 226]) be the natural lift of $f \in \mathrm{Diff}N$ to the bundle of metrics (resp. linear frame bundle); namely $f_M(g_x) = (f^{-1})^* g_x$ (resp. $\tilde{f}(X_1,\dots,X_N) = (f_*X_1,\dots,f_*X_N)$, where $(X_1,\dots,X_N) \in F_x(N)$); hence $p_M \circ f_M = f \circ p_M$ (resp. $p_F \circ \tilde{f} = f \circ p_F$), and $f_M \colon M \to M$ (resp. $\tilde{f} \colon F(N) \to F(N)$) have a natural extension to jet bundles $f_M^{(r)} \colon J^r(M) \to J^r(M)$ (resp. $\tilde{f}^{(r)} \colon J^r(FN) \to J^r(FN)$) as defined in the formula (3), i.e.,

$$
f_M^{(r)}(j_x^r g) = j_{f(x)}^r (f_M \circ g \circ f^{-1}) \quad (\text{resp. } \tilde{f}^{(r)}(j_x^r s) = j_{f(x)}^r(\tilde{f} \circ s \circ f^{-1})).
$$

As $\tilde{f}$ is an automorphism of the principal $Gl(n,\mathbb{R})$-bundle $F(N)$, it acts on linear connections by pulling back connection forms, i.e., $\Gamma' = \tilde{f}(\Gamma)$ where $\omega_{\Gamma'} = (\tilde{f}^{-1})^* \omega_\Gamma$ (see [**20**, II, Proposition 6.2-(b)], [**5**, 3.3]). Hence, there exists a unique diffeomorphism $\tilde{f}_C \colon C \to C$ such that,

1) $p_C \circ \tilde{f}_C = f \circ p_C$, and

2) $\tilde{f}_C \circ s_\Gamma = s_{\tilde{f}(\Gamma)}$ for every linear connection Γ.

If f_t is the flow of a vector field $X \in \mathfrak{X}(N)$, then the infinitesimal generator of $(f_t)_M$ (resp. $\tilde{f}_t$, resp. $(\tilde{f}_t)_C$) in $\mathrm{Diff}\, M$ (resp. $\mathrm{Diff}\, F(N)$, resp. $\mathrm{Diff}\, C$) is denoted by X_M (resp. $\tilde{X}$, resp. $\tilde{X}_C$) and the following Lie-algebra homomorphisms are obtained:

$$\begin{cases} \mathfrak{X}(N) \to \mathfrak{X}(M), & X \mapsto X_M \\ \mathfrak{X}(N) \to \mathfrak{X}(F(N)), & X \mapsto \tilde{X} \\ \mathfrak{X}(N) \to \mathfrak{X}(C), & X \mapsto \tilde{X}_C \end{cases}$$

If $X = u^i \partial/\partial x^i \in \mathfrak{X}(N)$ is the local expression for X, then

1) From [**30**, eqs. (2)–(4)] we know that the natural lift of X to M is given by,

$$X_M = u^i \frac{\partial}{\partial x^i} - \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M).$$

and its 1-jet prolongation,

$$\begin{aligned} X_M^{(1)} &= u^i \frac{\partial}{\partial x^i} - \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{hi} \right) \frac{\partial}{\partial y_{ij}} \\ &- \sum_{i \le j} \left(\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} + \frac{\partial u^h}{\partial x^i} y_{hj,k} + \frac{\partial u^h}{\partial x^j} y_{hi,k} + \frac{\partial u^h}{\partial x^k} y_{ij,h} \right) \frac{\partial}{\partial y_{ij,k}}. \end{aligned}$$

2) From [**10**, Proposition 3] (also see [**20**, VI, Proposition 21.1]) we know that the natural lift of X to $F(N)$ is given by,

$$\tilde{X} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x_j^l \frac{\partial}{\partial x_j^i},$$

and its 1-jet prolongation,

$$\begin{aligned} \tilde{X}^{(1)} &= u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x_j^l \frac{\partial}{\partial x_j^i} + v_{jk}^i \frac{\partial}{\partial x_{j,k}^i}, \\ v_{jk}^i &= \frac{\partial u^i}{\partial x^l} x_{j,k}^l - \frac{\partial u^l}{\partial x^k} x_{j,l}^i + \frac{\partial^2 u^i}{\partial x^k \partial x^l} x_j^l. \end{aligned}$$

3) Finally,

$$\tilde{X}_C = u^i \frac{\partial}{\partial x^i} - \left(\frac{\partial^2 u^i}{\partial x^j \partial x^k} - \frac{\partial u^i}{\partial x^l} A_{jk}^l + \frac{\partial u^l}{\partial x^k} A_{jl}^i + \frac{\partial u^l}{\partial x^j} A_{lk}^i \right) \frac{\partial}{\partial A_{jk}^i},$$

$$\tilde{X}_C^{(1)} = u^i \frac{\partial}{\partial x^i} + w^i_{jk} \frac{\partial}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial}{\partial A^i_{jk,h}},$$

$$(7) \qquad w^i_{jk} = -\frac{\partial^2 u^i}{\partial x^j \partial x^k} + \frac{\partial u^i}{\partial x^l} A^l_{jk} - \frac{\partial u^l}{\partial x^k} A^i_{jl} - \frac{\partial u^l}{\partial x^j} A^i_{lk},$$

$$(8) \qquad w^i_{jkh} = -\frac{\partial^3 u^i}{\partial x^h \partial x^j \partial x^k} + \frac{\partial^2 u^i}{\partial x^h \partial x^l} A^l_{jk} - \frac{\partial^2 u^l}{\partial x^h \partial x^k} A^i_{jl} - \frac{\partial^2 u^l}{\partial x^h \partial x^j} A^i_{lk} + \frac{\partial u^i}{\partial x^l} A^l_{jk,h} - \frac{\partial u^l}{\partial x^k} A^i_{jl,h} - \frac{\partial u^l}{\partial x^j} A^i_{lk,h} - \frac{\partial u^l}{\partial x^h} A^i_{jk,l}.$$

Let $p \colon M \times_N C \to N$ be the natural projection.

We denote by $\bar{f} = (f_M, \tilde{f}_C)$ (resp. $\bar{X} = (X_M, \tilde{X}_C) \in \mathfrak{X}(M \times_N C)$) the natural lift of f (resp. X) to $M \times_N C$. The prolongation to the bundle $J^1(M \times_N C)$ of $\bar{X}$ is as follows:

(9)

$$\bar{X}^{(1)} = \left(X_M^{(1)}, \tilde{X}_C^{(1)}\right) = u^i \frac{\partial}{\partial x^i} + \sum_{i \le j} v_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{i \le j} v_{ijk} \frac{\partial}{\partial y_{ij,k}} + w^i_{jk} \frac{\partial}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial}{\partial A^i_{jk,h}},$$

where

$$(10) \qquad v_{ij} = -\frac{\partial u^h}{\partial x^i} y_{hj} - \frac{\partial u^h}{\partial x^j} y_{hi},$$

$$(11) \qquad v_{ijk} = -\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} - \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} - \frac{\partial u^h}{\partial x^i} y_{hj,k} - \frac{\partial u^h}{\partial x^j} y_{hi,k} - \frac{\partial u^h}{\partial x^k} y_{ij,h},$$

and w^i_{jk}, w^i_{jkh} are given in the formulas (7), (8), respectively.

2.3 $\operatorname{Diff} N$- and $\mathfrak{X}(N)$-invariance

A differential form $\omega_r \in \Omega^r(J^1(M \times_N C))$, $r \in \mathbb{N}$, is said to be $\operatorname{Diff} N$-invariant— or invariant under diffeomorphisms— (resp. $\mathfrak{X}(N)$-invariant) if the following equation holds: $(\bar{f}^{(1)})^* \omega_r = \omega_r$, $\forall f \in \operatorname{Diff} N$ (resp. $L_{\bar{X}^{(1)}} \omega_r = 0$, $\forall X \in \mathfrak{X}(N)$). Obviously, "$\operatorname{Diff} N$-invariance" implies "$\mathfrak{X}(N)$-invariance" and the converse is almost true (see [**14**], [**28**]). Because of this, below we consider $\mathfrak{X}(N)$-invariance only.

A linear frame $(X_1, \dots, X_N)$ at x is said to be orthonormal with respect to $g_x \in M_x(N)$ (or simply g_x-orthonormal) if $g_x(X_i, X_j) = 0$ for $1 \le i < j \le n$, $g(X_i, X_i) = 1$ for $1 \le i \le n^+$, $g(X_i, X_i) = -1$ for $n^+ + 1 \le i \le n$.

As N is an oriented manifold, there exists a unique p-horizontal n-form $\mathbf{v}$ on $M\times_N C$ such that, $\mathbf{v}_{(g_x,\Gamma_x)}(X_1,\dots,X_N)=1$, for every g_x-orthonormal basis $(X_1,\dots,X_N)$ belonging to the orientation of N. Locally $\mathbf{v}=\rho v_n$, where $\rho=\sqrt{(-1)^{n^-}\det(y_{ij})}$ and $v_n=dx^1\wedge\dots\wedge dx^n$. As proved in [**30**, Proposition 7], the form $\mathbf{v}$ is Diff$N$-invariant and hence $\mathfrak{X}(N)$-invariant. A Lagrangian density Λ on $J^1(M\times_N C)$ can be globally written as $\Lambda=\mathcal{L}\mathbf{v}$ for a unique function $\mathcal{L}\in C^\infty(J^1(M\times_N C))$ and Λ is $\mathfrak{X}(N)$-invariant if and only if the function $\mathcal{L}$ is. Therefore, the invariance of Lagrangian densities is reduced to that of scalar functions.

Proposition 2.1. *A function $\mathcal{L}\in C^\infty(J^1(M\times_N C))$ is $\mathfrak{X}(N)$-invariant if and only if the following system of partial differential equations hold:*

$$
\begin{array}{ll}
0=X^i(\mathcal{L}), & \forall i,\\
0=X_h^i(\mathcal{L}), & \forall h,i,\\
0=X_h^{ik}(\mathcal{L}), & \forall h, i\leq k,\\
0=X_i^{jkh}(\mathcal{L}), & \forall i, j\leq k\leq h,
\end{array}
\tag{12}
$$

where

$$X^i=\frac{\partial}{\partial x^i},\ \forall i,$$

$$
\begin{aligned}
X_h^i={}&-y_{hi}\frac{\partial}{\partial y_{ii}}-y_{hj}\frac{\partial}{\partial y_{ij}}-y_{ih,k}\frac{\partial}{\partial y_{ii,k}}-y_{hj,k}\frac{\partial}{\partial y_{ij,k}}-\sum_{s\leq j}y_{sj,h}\frac{\partial}{\partial y_{sj,i}}\\
&+A_{jk}^i\frac{\partial}{\partial A_{jk}^h}-A_{jh}^r\frac{\partial}{\partial A_{ji}^r}-A_{hk}^r\frac{\partial}{\partial A_{ik}^r}\\
&+A_{jk,s}^i\frac{\partial}{\partial A_{jk,s}^h}-A_{jh,r}^s\frac{\partial}{\partial A_{ji,r}^s}-A_{hk,r}^s\frac{\partial}{\partial A_{ik,r}^s}-A_{jk,h}^r\frac{\partial}{\partial A_{jk,i}^r},\ \forall h,i,
\end{aligned}
$$

$$
\begin{aligned}
X_h^{ik}={}&-y_{ih}\frac{\partial}{\partial y_{ii,k}}-y_{kh}\frac{\partial}{\partial y_{kk,i}}-y_{hj}\frac{\partial}{\partial y_{ij,k}}-y_{hj}\frac{\partial}{\partial y_{kj,i}}-\frac{\partial}{\partial A_{ik}^h}-\frac{\partial}{\partial A_{ki}^h}\\
&+A_{js}^k\frac{\partial}{\partial A_{js,i}^h}-A_{jh}^s\frac{\partial}{\partial A_{jk,i}^s}-A_{hr}^s\frac{\partial}{\partial A_{kr,i}^s}\\
&+A_{js}^i\frac{\partial}{\partial A_{js,k}^h}-A_{jh}^s\frac{\partial}{\partial A_{ji,k}^s}-A_{hr}^s\frac{\partial}{\partial A_{ir,k}^s},\ \forall h,i\leq k,
\end{aligned}
\tag{13}
$$

$$
X_i^{jkh}=\frac{\partial}{\partial A_{jk,h}^i}+\frac{\partial}{\partial A_{jh,k}^i}+\frac{\partial}{\partial A_{hk,j}^i}+\frac{\partial}{\partial A_{hj,k}^i}+\frac{\partial}{\partial A_{kj,h}^i}+\frac{\partial}{\partial A_{kh,j}^i},\ \forall i,h\leq j\leq k.
\tag{14}
$$

Moreover, the vector fields $X^i, X_h^i, X_h^{ik}, X_i^{jkh}$ are linearly independent and they span an involutive distribution on $J^1(M\times_N C)$ of rank $n\binom{n+3}{3}$. Hence, the number of functionally invariant Lagrangians on $J^1(M\times_N C)$ is

$$\tfrac{1}{6}\left(5n^4+3n^3-5n^2+3n\right).$$

Proof. According to the formula (9), $\mathcal{L}$ is invariant if and only if,

$$u^i\frac{\partial\mathcal{L}}{\partial x^i}+\textstyle\sum_{i\leq j}v_{ij}\frac{\partial\mathcal{L}}{\partial y_{ij}}+\textstyle\sum_{i\leq j}v_{ijk}\frac{\partial\mathcal{L}}{\partial y_{ij,k}}+w^i_{jk}\frac{\partial\mathcal{L}}{\partial A^i_{jk}}+w^i_{jkh}\frac{\partial\mathcal{L}}{\partial A^i_{jk,h}}=0,$$
$$\forall u^i\in C^\infty(N),$$

and expanding on this equation by using the formulas (10), (11), (7), and (8) we obtain

$$\begin{aligned}
0 &= u^i\frac{\partial\mathcal{L}}{\partial x^i}\\
&+\frac{\partial u^h}{\partial x^i}\Bigg(-y_{hi}\frac{\partial\mathcal{L}}{\partial y_{ii}}-y_{hj}\frac{\partial\mathcal{L}}{\partial y_{ij}}-y_{ih,k}\frac{\partial\mathcal{L}}{\partial y_{ii,k}}-y_{hj,k}\frac{\partial\mathcal{L}}{\partial y_{ij,k}}\\
&-\sum_{s\leq j}y_{sj,h}\frac{\partial\mathcal{L}}{\partial y_{sj,i}}+A^i_{jk}\frac{\partial\mathcal{L}}{\partial A^h_{jk}}-A^r_{jh}\frac{\partial\mathcal{L}}{\partial A^r_{ji}}-A^r_{hk}\frac{\partial\mathcal{L}}{\partial A^r_{ik}}\\
&+A^i_{jk,s}\frac{\partial\mathcal{L}}{\partial A^h_{jk,s}}-A^s_{jh,r}\frac{\partial\mathcal{L}}{\partial A^s_{ji,r}}-A^s_{hk,r}\frac{\partial\mathcal{L}}{\partial A^s_{ik,r}}-A^r_{jk,h}\frac{\partial\mathcal{L}}{\partial A^r_{jk,i}}\Bigg)\\
&+\frac{\partial^2u^h}{\partial x^i\partial x^k}\Bigg(-y_{ih}\frac{\partial\mathcal{L}}{\partial y_{ii,k}}-y_{hj}\frac{\partial\mathcal{L}}{\partial y_{ij,k}}-\frac{\partial\mathcal{L}}{\partial A^h_{ik}}\\
&+A^k_{js}\frac{\partial\mathcal{L}}{\partial A^h_{js,i}}-A^s_{jh}\frac{\partial\mathcal{L}}{\partial A^s_{jk,i}}-A^r_{hr}\frac{\partial\mathcal{L}}{\partial A^r_{kr,i}}\Bigg)\\
&-\frac{\partial^3u^i}{\partial x^h\partial x^k\partial x^j}\frac{\partial\mathcal{L}}{\partial A^i_{jk,h}}.
\end{aligned}$$

This equation is equivalent to the system of the statement as the values for u^h, $\partial u^h/\partial x^i$, $\partial^2u^h/\partial x^i\partial x^j$ $(i\leq j)$, and $\partial^3u^h/\partial x^i\partial x^j\partial x^k$ $(i\leq j\leq k)$ at a point $x\in N$ can be taken arbitrarily. Moreover, assume a linear combination holds

$$\begin{aligned}&\lambda_aX^a+\lambda^a_bX^b_a+\textstyle\sum_{b\leq c}\lambda^a_{bc}X^{bc}_a+\textstyle\sum_{b\leq c\leq d}\lambda^a_{bcd}X^{bcd}_a=0,\\ &\lambda_a,\lambda^a_b,\lambda^a_{bc},\lambda^a_{bcd}\in C^\infty(J^1(M\times_NC)).\end{aligned}\tag{15}$$

By applying (15) to x^a (resp. y_{ab}) we obtain $\lambda_a=0$ (resp. $\lambda^a_b=0$); again by applying (15) to A^a_{bc}, $b\leq c$ (resp. A^a_{bc}, $c\leq b$) and taking the expressions of the vector fields (13) and (14) into account, we obtain $\lambda^a_{bc}=0$, $b\leq c$ (resp. $\lambda^a_{bc}=0$, $c\leq b$). Hence, (15) reads $\sum_{b\leq c\leq d}\lambda^a_{bcd}X^{bcd}_a=0$, and by applying it to $A^a_{bc,d}$ and taking the expressions of the vector fields (14) into account, we finally obtain $\lambda^a_{bcd}=0$. The distribution

$$\mathcal{D}_{M\times_NC}=\left\{\bar{X}^{(1)}_{(j^1_xg,j^1_xs_\Gamma)}:X\in\mathfrak{X}(N),\left(j^1_xg,j^1_xs_\Gamma\right)\in J^1(M\times_NC)\right\}$$

in $T\left(J^1(M\times_N C)\right)$, where $\bar{X}^{(1)}$ is defined in (9), is involutive as

$$\left[\bar{X}^{(1)},\bar{Y}^{(1)}\right]=\overline{[X,Y]}^{(1)},\quad \forall X,Y\in\mathfrak{X}(N),$$

and it is spanned by $X^i, X^i_h, X^{ik}_h, X^{jkh}_i$, as proved by the formulas above. The rest of the statement follows from the following identities:

$$\begin{aligned}&\#\left\{X^i;X^i_h;X^{ik}_h,i\leq k;X^{jkh}_i,h\leq j\leq k:h,i,j,k=1,\dots,n\right\}\\ &\qquad\qquad\qquad\qquad=n+n^2+n\tbinom{n+1}{2}+n\tbinom{n+2}{3}=n\tbinom{n+3}{3},\\ &\qquad\dim J^1\left(M\times_N C\right)-n\tbinom{n+3}{3}=\tfrac{1}{6}\left(5n^4+3n^3-5n^2+3n\right).\end{aligned}$$

□

3 Invariance of covariant Hamiltonians

3.1 Position of the problem

On the bundle $E=M\times_N C$, an Ehresmann connection can locally be written as follows:

$$\begin{aligned}&\gamma=\textstyle\sum_{i\leq j}\left(dy_{ij}+\gamma_{ijk}dx^k\right)\otimes\dfrac{\partial}{\partial y_{ij}}+\left(dA^i_{jk}+\gamma^i_{jkl}dx^l\right)\otimes\dfrac{\partial}{\partial A^i_{jk}},\\ &\gamma_{ijk},\gamma^i_{jkl}\in C^\infty(M\times_N C).\end{aligned}\tag{16}$$

In particular, for a Lagrangian density Λ on $J^1(M\times_N C)$ we obtain

$$\Lambda^\gamma=\left(\sum_{i\leq j}\left(\gamma_{ijk}+y_{ij,k}\right)\frac{\partial L}{\partial y_{ij,k}}+\left(\gamma^i_{jkl}+A^i_{jk,l}\right)\frac{\partial L}{\partial A^i_{jk,l}}-L\right)dx^1\wedge\cdots\wedge dx^n,$$

or equivalently, $\mathcal{L}^\gamma=D^\gamma(\mathcal{L})-\mathcal{L}$, where

$$D^\gamma=\sum_{i\leq j}\left(\gamma_{ijk}+y_{ij,k}\right)\frac{\partial}{\partial y_{ij,k}}+\left(\gamma^i_{jkl}+A^i_{jk,l}\right)\frac{\partial}{\partial A^i_{jk,l}}.$$

Remark 3.1. The horizontal form $(p^1_0)^*\gamma-\theta=\left(\gamma^\alpha_i+y^\alpha_i\right)dx^i\otimes\partial/\partial y^\alpha$ can also be viewed as the p^1_0-vertical vector field

$$D^\gamma=\left(\gamma^\alpha_i+y^\alpha_i\right)\frac{\partial}{\partial y^\alpha_i},\tag{17}$$

taking the natural isomorphism $V(p^1_0)\cong(p^1_0)^*(p^*T^*N\otimes V(p))$ into account (cf. [**23**], [**24**], [**32**], [**34**]).

According to the previous formulas, this means: If the system (12) holds for a Lagrangian function $\mathcal{L}$, then it also holds for the covariant Hamiltonian $\mathcal{L}^\gamma$.

If $X \in \{X^i, X^i_h, X^{ik}_h, X^{jkh}_i\}$, then $X(\mathcal{L}^\gamma) = X(D^\gamma(\mathcal{L}))$, as $\mathcal{L}$ is assumed to be invariant and hence $X(\mathcal{L}) = 0$. Therefore

$$\begin{aligned} X(\mathcal{L}^\gamma) &= X(D^\gamma(\mathcal{L})) \\ &= [X, D^\gamma](\mathcal{L}), \end{aligned}$$

and we conclude the following:

Proposition 3.2. *The property* **(P)** *holds for an Ehresmann connection γ on $M \times_N C$ if and only if the vector field D^γ transforms the sections of the distribution $\mathcal{D}_{M\times_N C}$ into themselves, namely, $[D^\gamma, \Gamma(\mathcal{D}_{M\times_N C})] \subseteq \Gamma(\mathcal{D}_{M\times_N C})$.*

The problem thus reduces to compute the brackets $\left[X^i, D^\gamma\right]$, $\left[X^i_h, D^\gamma\right]$, $\left[X^{ik}_h, D^\gamma\right]$, and $[X^{jkh}_i, D^\gamma]$. We have

$$\left[X^h, D^\gamma\right] = \sum_{i\le j} \frac{\partial \gamma_{ijk}}{\partial x^h}\frac{\partial}{\partial y_{ij,k}} + \frac{\partial \gamma^i_{jkl}}{\partial x^h}\frac{\partial}{\partial A^i_{jk,l}}, \tag{18}$$

$$\left[X^{cda}_b, D^\gamma\right] = X^{cda}_b, \quad \forall b, c \le d \le a,$$

$$\begin{aligned} \left[X^i_h, D^\gamma\right] = {} & \sum_{a\le b} Y^i_h(\gamma_{abk})\frac{\partial}{\partial y_{ab,k}} + \sum_{i\le h}\gamma_{ihk}\frac{\partial}{\partial y_{ii,k}} + \sum_{h<i}\gamma_{hik}\frac{\partial}{\partial y_{ii,k}} \\ & + \sum_{h\le j}\gamma_{hjk}\frac{\partial}{\partial y_{ij,k}} + \sum_{j<h}\gamma_{jhk}\frac{\partial}{\partial y_{ij,k}} + \sum_{a\le b}\gamma_{abh}\frac{\partial}{\partial y_{ab,i}} \\ & + \left(Y^i_h(\gamma^a_{bcr}) - \delta^h_a\gamma^i_{bcr} + \delta^c_i\gamma^a_{bhr} + \delta^b_i\gamma^a_{hcr} + \delta^r_i\gamma^a_{bch}\right)\frac{\partial}{\partial A^a_{bc,r}}, \end{aligned} \tag{19}$$

$$\left[X^{ik}_h, D^\gamma\right] = \sum_{a\le b} Y^{ik}_h(\gamma_{abc})\frac{\partial}{\partial y_{ab,c}} + Y^{ik}_h\left(\gamma^d_{abc}\right)\frac{\partial}{\partial A^d_{ab,c}} + X^{ik}_h - Y^{ik}_h, \tag{20}$$

where

$$Y^i_h = -y_{hi}\frac{\partial}{\partial y_{ii}} - y_{hj}\frac{\partial}{\partial y_{ij}} + A^i_{jk}\frac{\partial}{\partial A^h_{jk}} - A^r_{jh}\frac{\partial}{\partial A^r_{ji}} - A^r_{hk}\frac{\partial}{\partial A^r_{ik}},$$

$$Y^{ik}_h = -\frac{\partial}{\partial A^h_{ik}} - \frac{\partial}{\partial A^h_{ki}},$$

and the following formula has been used:

$$\frac{\partial y_{rs,k}}{\partial y_{ij,h}} = \delta_h^k \left(\delta_i^r \delta_j^s + \delta_j^r \delta_i^s - \delta_j^i \delta_r^i \delta_s^j\right).$$

3.2 The class of the Ehresmann connections defined

Let $p\colon M \times_N C \to N$, $\mathrm{pr}_1\colon M \times_N C \to M$, $\mathrm{pr}_2\colon M \times_N C \to C$ be the natural projections. By taking the differential of pr_1 and pr_2, a natural identification is obtained $T(M \times_N C) = TM \times_{TN} TC$. Hence

$$\begin{aligned} V(p) &= V(p_M) \times_N V(p_C) \\ &= \mathrm{pr}_1^* V(p_M) \oplus \mathrm{pr}_2^* V(p_C) \end{aligned}$$

and two unique vector-bundle homomorphisms exist

$$\gamma_M\colon \mathrm{pr}_1^* TM \to \mathrm{pr}_1^* V(p_M), \quad \gamma_C\colon \mathrm{pr}_2^* TC \to \mathrm{pr}_2^* V(p_C),$$

such that,

$$\begin{array}{ll} \gamma(X) = \left(\gamma_M\left(\mathrm{pr}_{1*}X\right), \gamma_C\left(\mathrm{pr}_{2*}X\right)\right), & \forall X \in T(M \times_N C), \\ \gamma_M(Y) = Y, & \forall Y \in \mathrm{pr}_1^* V(p_M), \\ \gamma_C(Z) = Z, & \forall Z \in \mathrm{pr}_2^* V(p_C). \end{array}$$

If γ is given by the local expression of the formula (16), then

$$\gamma_M = \textstyle\sum_{i\le j}\left(dy_{ij} + \gamma_{ijk} dx^k\right) \otimes \dfrac{\partial}{\partial y_{ij}}, \ \gamma_C = \left(dA_{jk}^i + \gamma_{jkl}^i dx^l\right) \otimes \dfrac{\partial}{\partial A_{jk}^i},$$
$$\gamma_{ijk}, \gamma_{jkl}^i \in C^\infty(M \times_N C).$$

3.2.1 The first geometric condition on γ

Let $q\colon F(N) \to M$ be the projection given by

$$\begin{aligned} q(X_1, \dots, X_N) &= g_x \\ &= \varepsilon_h w^h \otimes w^h, \end{aligned} \tag{21}$$

where $(w^1, \dots, w^n)$ is the dual coframe of $(X_1, \dots, X_N) \in F_x(N)$, i.e., g_x is the metric for which $(X_1, \dots, X_N)$ is a g_x-orthonormal basis and $\varepsilon_h = 1$ for $1 \le h \le n^+$, $\varepsilon_h = -1$ for $n^+ + 1 \le h \le n$. As readily seen, q is a principal G-bundle with $G = O(n^+, n^-)$.

Given a linear connection Γ and a tangent vector $X \in T_x N$, for every u in $p^{-1}(x)$ there exists a unique Γ-horizontal tangent vector $X_u^{h\Gamma} \in T_u(FN)$

such that, $(p_F)_* X_u^{h_\Gamma} = X$. The local expression for the horizontal lift is known to be ([**20**, Chapter III, Proposition 7.4]),

$$\left(\frac{\partial}{\partial x^j}\right)^{h_\Gamma} = \frac{\partial}{\partial x^j} - \Gamma^i_{jk} x^k_l \frac{\partial}{\partial x^i_l}. \tag{22}$$

Lemma 3.3. *Given a metric $g_x \in p_M^{-1}(x)$, let $u \in p_F^{-1}(x)$ be a linear frame such that $q(u) = g_x$. The projection $q_*(X_u^{h_{\Gamma_x}})$ does not depend on the linear frame u chosen over g_x.*

Proof. In fact, any other linear frame projecting onto g_x can be written as $u \cdot A$, $A \in G$. As the horizontal distribution is invariant under right translations (see [**20**, II, Proposition 1.2]), the following equation holds: $(R_A)_* \left(X_u^{h_\Gamma}\right) = X_{u\cdot A}^{h_\Gamma}$. Hence

$$\begin{aligned} q_*\left(X_{u\cdot A}^{h_\Gamma}\right) &= q_*\left((R_A)_*\left(X_u^{h_\Gamma}\right)\right) \\ &= (q \circ R_A)_*\left(X_u^{h_\Gamma}\right) \\ &= q_*\left(X_u^{h_\Gamma}\right). \end{aligned}$$

□

Proposition 3.4. *An Ehresmann connection γ on $M \times_N C$ satisfies the following condition:*

$$(C_M)\text{:}\ \gamma_M\left((g_x, \Gamma_x), X\right) = X - q_*\left(\left((p_M)_*(X)\right)_u^{h_{\Gamma_x}}\right),$$

$\forall X \in T_{g_x}M$, $u \in q^{-1}(g_x)$, (which does not depend on the linear frame $u \in q^{-1}(g_x)$ chosen, according to Lemma 3.3*) if and only if the following equations hold:*

$$\gamma_{klj} = -\left(y_{al} A^a_{jk} + y_{ak} A^a_{jl}\right), \tag{23}$$

where the functions γ_{klj} (resp. y_{ij}, resp. A^i_{jk}) are defined in the formula (16) *(resp.* (4)*, resp.* (6)*).*

Proof. Letting $(\chi^i_j)^n_{i,j=1} = \left((x^i_j)^n_{i,j=1}\right)^{-1}$, the dual coframe of the linear frame $u = (X_1, \dots, X_N) \in F_x(N)$ given in (5) is $(w^1, \dots, w^n)$, $w^h = \chi^h_k(u)\left(dx^k\right)_x$, $1 \le h \le n$, and the projection q is given by

$$\begin{aligned} q(u) &= g_x \\ &= \sum\nolimits_{h=1}^n \varepsilon_h \chi^h_k(u) \chi^h_l(u) \left(dx^k\right)_x \otimes \left(dx^l\right)_x. \end{aligned}$$

Therefore the equations of the projection (21) are as follows:

$$\begin{aligned} x^i \circ q &= x^i, \\ y_{kl} \circ q &= \textstyle\sum_{h=1}^n \varepsilon_h \chi_k^h \chi_l^h. \end{aligned}$$

Hence

$$q_* \left(\frac{\partial}{\partial x_b^a} \right)_u = \sum_{k \leq l} \varepsilon_h \left\{ \frac{\partial \chi_k^h}{\partial x_b^a} \chi_l^h + \chi_k^h \frac{\partial \chi_l^h}{\partial x_b^a} \right\} (u) \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}.$$

Taking derivatives with respect to x_b^a on the identity $\chi_r^h x_i^r = \delta_i^h$, multiplying the outcome by χ_k^i, and summing up over the index i, the following formula is obtained: $\partial \chi_k^h / \partial x_b^a = -\chi_a^h \chi_k^b$. Replacing this equation into the expression for $q_* \left(\partial / \partial x_b^a \right)_u$ above, we have

$$q_* \left(\frac{\partial}{\partial x_b^a} \right)_u = -\sum_{k \leq l} \left\{ \chi_k^b(u) y_{al} \left(g_x \right) + \chi_l^b(u) y_{ak} \left(g_x \right) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}.$$

From (22), evaluated at $u \in q^{-1}(g_x)$, we deduce

$$\begin{aligned} q_* \left(\frac{\partial}{\partial x^j} \right)_u^{h_\Gamma} &= \left(\frac{\partial}{\partial x^j} \right)_{g_x} - \Gamma_{jc}^a(x) x_b^c(u) q_* \left(\frac{\partial}{\partial x_b^a} \right)_{g_x} \\ &= \left(\frac{\partial}{\partial x^j} \right)_{g_x} \\ &\quad + \textstyle\sum_{k \leq l} \Gamma_{jc}^a(x) x_b^c(u) \left\{ \chi_k^b(u) y_{al} \left(g_x \right) + \chi_l^b(u) y_{ak} \left(g_x \right) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x} \\ &= \left(\frac{\partial}{\partial x^j} \right)_{g_x} + \textstyle\sum_{k \leq l} \left\{ \Gamma_{jk}^a(x) y_{al} \left(g_x \right) + \Gamma_{jl}^a(x) y_{ak} \left(g_x \right) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}. \end{aligned}$$

The condition (C_M) holds automatically whenever $X \in V(p_M)$. Hence, (C_M) holds if and only if it holds for $X = (\partial / \partial x^j)_{g_x}$, namely,

$$\begin{aligned} \sum_{k \leq l} \gamma_{klj}(g_x, \Gamma_x) \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x} &= \gamma_M \left(\left(g_x, \Gamma_x \right), \left(\frac{\partial}{\partial x^j} \right)_{g_x} \right) \\ &= \left(\frac{\partial}{\partial x^j} \right)_{g_x} - q_* \left(\frac{\partial}{\partial x^j} \right)_u^{h_{\Gamma_x}} \\ &= -\sum_{k \leq l} \left\{ \Gamma_{jk}^a(x) y_{al} \left(g_x \right) + \Gamma_{jl}^a(x) y_{ak} \left(g_x \right) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}, \end{aligned}$$

thus proving the formula (23) in the statement. □

3.2.2 The canonical covariant derivative

As is known (e.g., see [**20**, III, section 1], [**23**, pp. 157–158]) every connection Γ on a principal G-bundle $P \to N$ induces a covariant derivative ∇^Γ on the vector bundle associated to P under a linear representation $\rho\colon G \to Gl(m,\mathbb{R})$ with standard fibre $\mathbb{R}^m$. In particular, this applies to the principal bundle of linear frames, thus proving that every linear connection Γ on N induces a covariant derivative ∇^Γ on every tensorial vector bundle $E \to N$.

The bundles $(p_C)^*E$, where E is a tensorial vector bundle, are endowed with a canonical covariant derivative ∇^E completely determined by the formula:

$$\left(\left(\nabla^E\right)_X (f\xi)\right)(\Gamma_x) = \left(\left(Xf\right)\xi\right)(\Gamma_x) + f\left(\Gamma_x\right)\left(\nabla^{\Gamma_x}_{(p_C)_*X}\xi\right)(x), \tag{24}$$

for all $X \in T_{\Gamma_x}C$, $f \in C^\infty(C)$, and every local section ξ of E defined on a neighbourhood of x. The uniqueness of ∇^E follows from (24) as the sections of E span the sections of $(p_C)^*E$ over $C^\infty(C)$, see [**8**, 0.3.6]. Below, we are specially concerned with the cases $E = TN$ and $E = \wedge^2 T^*N \otimes TN$.

3.2.3 The 2-form associated with γ_C

As $p_C\colon C \to N$ is an affine bundle modelled over $\otimes^2 T^*N \otimes TN$, there is a natural identification

$$V(p_C) \cong (p_C)^*\left(\otimes^2 T^*N \otimes TN\right)$$

and consequently, an Ehresmann connection γ_C on C can also be viewed as a homomorphism $\gamma_C\colon TC \to \otimes^2 T^*N \otimes TN$. If γ_C is locally given by

$$\gamma_C = \left(dA^i_{jk} + \gamma^i_{jkl}dx^l\right) \otimes \frac{\partial}{\partial A^i_{jk}}, \quad \gamma^i_{jkl} \in C^\infty(C), \tag{25}$$

then

$$\gamma_C = (dA^i_{jk} + \gamma^i_{jkl}dx^l) \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i},$$

and γ_C induces a 2-form $\tilde{\gamma}_C$ taking values in $(p_C)^*(T^*N \otimes TN)$ as follows:

$$\begin{aligned}&\tilde{\gamma}_C(X,Y) = c^1_1\left((p_C)_*(Y) \otimes \gamma_C(X)\right) - c^1_1\left((p_C)_*(X) \otimes \gamma_C(Y)\right),\\ &\forall X, Y \in T_{\Gamma_x}C,\end{aligned}$$

where

$$\begin{aligned}&c^1_1\colon TN \otimes T^*N \otimes T^*N \otimes TN \to T^*N \otimes TN,\\ &c^1_1\left(X_1 \otimes w_1 \otimes w_2 \otimes X_2\right) = w_1(X_1)w_2 \otimes X_2,\\ &X_1, X_2 \in T_xN,\ w_1, w_2 \in T^*_xN.\end{aligned}$$

If γ_C is given by (25), then from the very definition of $\tilde{\gamma}_C$ the following local expression is obtained:

$$\tilde{\gamma}_C = (dA^c_{lh} + (\gamma^c_{lha} - \gamma^c_{ahl})\, dx^a) \wedge dx^l \otimes dx^h \otimes \frac{\partial}{\partial x^c}.$$

3.2.4 The second geometric condition on γ

Let $\mathrm{alt}_{12}\colon \otimes^2 T^*N \otimes TN \to \wedge^2 T^*N \otimes TN$ be the operator alternating the two covariant arguments.

The vector bundle $(p_C)^* \left(\wedge^2 T^*N \otimes TN\right)$ admits a canonical section

$$\tau_N \colon C \to \wedge^2 T^*N \otimes TN,$$
$$\tau_N(\Gamma_x) = T^{\Gamma_x},\ \forall \Gamma_x \in C,$$

where T^{Γ_x} is the torsion of Γ_x. Locally,

$$\tau_N = \sum_{j<k} (A^i_{jk} - A^i_{kj}) dx^j \wedge dx^k \otimes \frac{\partial}{\partial x^i}.$$

From the previous formulas the next result follows:

Proposition 3.5. *Let γ be an Ehresmann connection on $M \times_N C$, let $\nabla^{(1)} = \nabla^{E_1}$ with $E_1 = TN$, let $R^{\nabla^{(1)}}$ be its curvature form, and finally, let $\nabla^{(2)} = \nabla^{E_2}$ with $E_2 = \wedge^2 T^*N \otimes TN$.*

(C_C) *Assume the component γ_C of γ is defined on C. Then, the equations*

$$\tilde{\gamma}_C = R^{\nabla^{(1)}}, \tag{26}$$

$$\mathrm{alt}_{12} \circ \gamma_C = \nabla^{(2)} \tau_N, \tag{27}$$

are locally equivalent to the following ones:

$$\gamma^h_{str} - \gamma^h_{rts} = A^h_{rm} A^m_{st} - A^h_{sm} A^m_{rt}, \tag{28}$$

$$\begin{aligned} \gamma^h_{rst} - \gamma^h_{srt} = A^h_{tm} \left(A^m_{rs} - A^m_{sr}\right) + A^m_{ts} \left(A^h_{mr} - A^h_{rm}\right) \\ + A^m_{tr} \left(A^h_{sm} - A^h_{ms}\right). \end{aligned} \tag{29}$$

3.3 Solution to the problem (P)

Theorem 3.6. *If the connection γ on $M \times_N C$ satisfies the conditions (C_M) and (C_C) introduced above, then the vector field D^γ satisfies the property stated in* Proposition 3.2 *and, accordingly the covariant Hamiltonian with respect to γ of every $\mathfrak{X}(N)$-invariant Lagrangian is also $\mathfrak{X}(N)$-invariant.*

Proof. When γ_M satisfies the condition (C_M) the brackets (18), (19), and (20) are respectively given by

$$\left[X^h, D^\gamma\right] = \frac{\partial \gamma^i_{jkl}}{\partial x^h} \frac{\partial}{\partial A^i_{jk,l}}, \tag{30}$$

$$\left[X^i_h, D^\gamma\right] = \left(Y^i_h\left(\gamma^a_{bcr}\right) - \delta^h_a \gamma^i_{bcr} + \delta^c_i \gamma^a_{bhr} + \delta^b_i \gamma^a_{hcr} + \delta^r_i \gamma^a_{bch}\right) \frac{\partial}{\partial A^a_{bc,r}}, \tag{31}$$

$$\begin{aligned}\left[X^{ik}_h, D^\gamma\right] = \Bigg(&-\frac{\partial \gamma^d_{abc}}{\partial A^h_{ik}} + \delta^c_i\left(\delta^h_d A^k_{ab} - \delta^k_b A^d_{ah} - \delta^k_a A^d_{hb}\right) \\ &-\frac{\partial \gamma^d_{abc}}{\partial A^h_{ki}} + \delta^c_k\left(\delta^h_d A^i_{ab} - \delta^i_b A^d_{ah} - \delta^i_a A^d_{hb}\right)\Bigg) \frac{\partial}{\partial A^d_{ab,c}}.\end{aligned}$$

In addition, if γ_C satisfies the condition (C_C), then taking derivatives with respect to x^h in (28) and (29) we obtain

$$\frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{jlk}}{\partial x^h}, \quad \frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{kjl}}{\partial x^h},$$

and renaming indices we deduce

$$\begin{aligned}&\frac{\partial \gamma^i_{jjk}}{\partial x^h} = \frac{\partial \gamma^i_{jkj}}{\partial x^h} = \frac{\partial \gamma^i_{kjj}}{\partial x^h} \ (j<k),\\ &\frac{\partial \gamma^i_{kkj}}{\partial x^h} = \frac{\partial \gamma^i_{kjk}}{\partial x^h} = \frac{\partial \gamma^i_{jkk}}{\partial x^h} \ (j<k),\\ &\frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{ljk}}{\partial x^h} = \frac{\partial \gamma^i_{kjl}}{\partial x^h} = \frac{\partial \gamma^i_{lkj}}{\partial x^h} = \frac{\partial \gamma^i_{jlk}}{\partial x^h} \ (j<k<l).\end{aligned}$$

From (30) we obtain

$$\begin{aligned}\left[X^h, D^\gamma\right] = &\sum_{j<k<l} \frac{\partial \gamma^i_{jkl}}{\partial x^h} X^{jkl}_i + \tfrac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{jjk}}{\partial x^h} X^{jjk}_i \\ &+ \tfrac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{kkj}}{\partial x^h} X^{kkj}_i + \tfrac{1}{6} \frac{\partial \gamma^i_{jjj}}{\partial x^h} X^{jjj}_i,\end{aligned}$$

and consequently the values of $\left[X^h, D^\gamma\right]$ belong to the distribution $\mathcal{D}_{M \times_N C}$.

Moreover, as γ_C is assumed to be defined on C, we have

$$Y^i_h\left(\gamma^a_{bcr}\right) = \left(\delta^s_h A^i_{jk} - \delta^i_k A^s_{jh} - \delta^i_j A^s_{hk}\right) \frac{\partial \gamma^a_{bcr}}{\partial A^s_{jk}}.$$

For the sake of simplicity, below we set

$$\left(T^i_h\right)^a_{bcr} = A^i_{jk} \frac{\partial \gamma^a_{bcr}}{\partial A^h_{jk}} - A^s_{jh} \frac{\partial \gamma^a_{bcr}}{\partial A^s_{ji}} - A^s_{hk} \frac{\partial \gamma^a_{bcr}}{\partial A^s_{ik}} - \delta^h_a \gamma^i_{bcr} + \delta^b_i \gamma^a_{hcr} + \delta^c_i \gamma^a_{bhr} + \delta^r_i \gamma^a_{bch}.$$

Taking derivatives with respect to A^s_{jk}, the equations (28) y (29) yield

$$\frac{\partial\gamma^a_{bcr}}{\partial A^s_{jk}} - \frac{\partial\gamma^a_{rcb}}{\partial A^s_{jk}} = \delta^j_r\delta^a_s A^k_{bc} - \delta^j_b\delta^a_s A^k_{rc} + \delta^j_b\delta^k_C A^a_{rs} - \delta^j_r\delta^k_C A^a_{bs},$$

$$\begin{aligned}\frac{\partial\gamma^a_{rbc}}{\partial A^s_{jk}} - \frac{\partial\gamma^a_{brc}}{\partial A^s_{jk}} = {} & \delta^j_C\delta^a_s A^k_{rb} - \delta^a_s\delta^j_C A^k_{br} - \delta^a_s\delta^k_b A^j_{cr} - \delta^a_s\delta^j_r A^k_{cb} + \delta^a_s\delta^k_r A^j_{cb} + \delta^a_s\delta^j_b A^k_{cr} \\ & + \delta^j_C\delta^k_b A^a_{sr} - \delta^j_C\delta^k_r A^a_{sb} + \delta^j_r\delta^k_b A^a_{cs} - \delta^j_b\delta^k_r A^a_{cs} + \delta^j_C\delta^k_r A^a_{bs} - \delta^j_C\delta^k_b A^a_{rs}.\end{aligned}$$

From these expressions, the following symmetries of indices are obtained:

$$\begin{aligned}&\left(T^i_h\right)^a_{bbc} = \left(T^i_h\right)^a_{bcb} = \left(T^i_h\right)^a_{cbb} \ (b<c),\\ &\left(T^i_h\right)^a_{bcc} = \left(T^i_h\right)^a_{cbc} = \left(T^i_h\right)^a_{ccb} \ (b<c),\\ &\left(T^i_h\right)^a_{bcd} = \left(T^i_h\right)^a_{dbc} = \left(T^i_h\right)^a_{cdb} = \left(T^i_h\right)^a_{bdc} = \left(T^i_h\right)^a_{dcb} = \left(T^i_h\right)^a_{cbd} \ (b<c<d),\end{aligned}$$

and from (31) we obtain

$$\begin{aligned}\left[X^i_h, D^\gamma\right] = {} & \sum_{b<c<d}\left(T^i_h\right)^a_{bcd} X^{bcd}_a + \tfrac{1}{2}\sum_{b<c}\left(T^i_h\right)^a_{bbc} X^{bbc}_a \\ & + \tfrac{1}{2}\sum_{b<c}\left(T^i_h\right)^a_{ccb} X^{ccb}_a + \tfrac{1}{6}\left(T^i_h\right)^a_{bbb} X^{bbb}_a.\end{aligned}$$

Hence $\left[X^i_h, D^\gamma\right]$ also takes values into the distribution $\mathcal{D}_{M\times_N C}$.

The proof for the third bracket is similar to the previous two cases but longer. Letting

$$\begin{aligned}\left(T^{ik}_h\right)^a_{rbc} = -\frac{\partial\gamma^a_{rbc}}{\partial A^h_{ik}} - \frac{\partial\gamma^a_{rbc}}{\partial A^h_{ki}} & + \delta^c_i\left(\delta^h_a A^k_{rb} - \delta^k_b A^a_{rh} - \delta^k_r A^a_{hb}\right) \\ & + \delta^c_k\left(\delta^h_a A^i_{rb} - \delta^i_b A^a_{rh} - \delta^i_r A^a_{hb}\right),\end{aligned}$$

the following symmetries are obtained:

$$\begin{aligned}&\left(T^{ik}_h\right)^a_{bbc} = \left(T^{ik}_h\right)^a_{bcb} = \left(T^{ik}_h\right)^a_{cbb} (b<c),\\ &\left(T^{ik}_h\right)^a_{bcc} = \left(T^{ik}_h\right)^a_{cbc} = \left(T^{ik}_h\right)^a_{ccb} (b<c),\\ &\left(T^{ik}_h\right)^a_{bcd} = \left(T^{ik}_h\right)^a_{dbc} = \left(T^{ik}_h\right)^a_{cdb} = \left(T^{ik}_h\right)^a_{bdc} = \left(T^{ik}_h\right)^a_{dcb} = \left(T^{ik}_h\right)^a_{cbd} (b<c<d).\end{aligned}$$

Hence

$$\begin{aligned}\left[X^{ik}_h, D^\gamma\right] = {} & \sum_{b<c<d}\left(T^{ik}_h\right)^a_{bcd} X^{bcd}_a + \tfrac{1}{2}\sum_{b<c}\left(T^{ik}_h\right)^a_{bbc} X^{bbc}_a \\ & + \tfrac{1}{2}\sum_{b<c}\left(T^{ik}_h\right)^a_{ccb} X^{ccb}_a + \tfrac{1}{6}\left(T^{ik}_h\right)^a_{bbb} X^{bbb}_a,\end{aligned}$$

and the proof is complete. $\square$

Theorem 3.7. *The Ehresmann connections on C satisfying the equations (26) and (27) are the sections of an affine bundle over C modelled over the vector bundle $(p_C)^* \left(S^3T^*N \otimes TN\right)$. Consequently, there always exist Ehresmann connections on $M \times_N C$ fulfilling the conditions (C_M) and (C_C) introduced above.*

Proof. If two Ehresmann connections γ_C, γ'_C satisfy the equations (26) and (27), then the difference tensor field $t = \gamma'_C - \gamma_C$, which is a section of the bundle $(p_C)^* \left(\otimes^3 T^*N \otimes TN\right)$, satisfies the following symmetries:

$$t(X_1, X_2, X_3) = t(X_3, X_2, X_1), \tag{32}$$

$$t(X_1, X_2, X_3) = t(X_2, X_1, X_3), \tag{33}$$

according to (28), (29), respectively, for all $X_1, X_2, X_3 \in T_xN$, $\Gamma_x \in C_x(N)$. Hence

$$t(X_1, X_3, X_2) \overset{(32)}{=} t(X_2, X_3, X_1) \overset{(33)}{=} t(X_3, X_2, X_1) \overset{(32)}{=} t(X_1, X_2, X_3),$$

thus proving that t is totally symmetric. The second part of the statement thus follows from the fact that an affine bundle always admits global sections, e.g., see [**20**, I, Theorem 5.7]. □

Remark 3.8. The results obtained above also hold if the bundle of linear connections is replaced by the subbundle $C^{\mathrm{sym}} = C^{\mathrm{sym}}(N) \subset C$ of symmetric linear connections; the only difference to be observed between both bundles is that in the symmetric cases the equation (27), or equivalently (29), holds automatically.

4 The second-order formalism

In this section we consider the problem of invariance of covariant Hamiltonians for second-order Lagrangians defined on the bundle of metrics, i.e., for functions $\mathcal{L} \in C^\infty(J^2M)$, where M denotes, as throughout this paper, the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-) on N.

4.1 Second-order Ehresmann connections

A second-order Ehresmann connection on $p \colon E \to N$ is a differential 1-form γ^2 on J^1E taking values in the vertical sub-bundle $V(p^1)$ such that $\gamma^2(X) = X$ for every $X \in V(p^1)$. (We refer the reader to [**29**] for the basics on Ehresmann connections of arbitrary order.) Once a connection γ^2 is given, we have a decomposition of vector bundles $T(J^1E) = V(p^1) \oplus \ker \gamma^2$, where $\ker \gamma^2$ is called the horizontal sub-bundle determined by γ^2. In the coordinate system on J^1E induced from a fibred coordinate system (x^j, y^α) for p, a connection form can be written as

$$\gamma^2 = (dy^\alpha + \gamma_j^\alpha dx^j) \otimes \frac{\partial}{\partial y^\alpha} + (dy_i^\alpha + \gamma_{ij}^\alpha dx^j) \otimes \frac{\partial}{\partial y_i^\alpha}, \quad \gamma_j^\alpha, \gamma_{ij}^\alpha \in C^\infty(J^1E). \tag{34}$$

As in the first-order case, the action of the group $\mathrm{Aut}(p)$ on the space of second-order connections is defined by the formula

$$\Phi \cdot \gamma^2 = \left(\Phi^{(1)}\right)_* \circ \gamma^2 \circ \left(\Phi^{(1)}\right)_*^{-1}, \quad \forall \Phi \in \mathrm{Aut}(p).$$

As $\Phi^{(1)} \colon J^1M \to J^1M$ is a morphism of fibred manifolds over N, $(\Phi^{(1)})_*$ transforms the vertical subbundle $V(p^1)$ into itself; hence the previous definition makes sense.

4.2 A remarkable isomorphism

Theorem 4.1. *Let Γ^g be the Levi-Civita connection of a pseudo-Riemannian metric g on N. The mapping $\zeta_N \colon J^1M \to M \times_N C^{\mathrm{sym}}$, $\zeta_N(j_x^1 g) = (g_x, \Gamma_x^g)$ is a diffeomorphism. There is a natural one-to-one correspondence between first-order Ehresmann connections on the bundle $p \colon M \times_N C^{\mathrm{sym}} \to N$ and second-order Ehresmann connections on the bundle $p_M \colon M \to N$, which is explicitly given by,*

$$\gamma^2 = ((\zeta_N^v)_*)^{-1} \circ \gamma \circ (\zeta_N)_*\,, \tag{35}$$

where $\gamma \colon T(M \times_N C^{\mathrm{sym}}) \to V(p)$ is a first-order Ehresmann connection,

$$(\zeta_N)_* : T(J^1M) \to T(M \times_N C^{\mathrm{sym}})$$

is the Jacobian mapping induced by ζ_N, and $(\zeta_N^v)_* : V(p_M^1) \to V(p)$ is its restriction to the vertical bundles.

Proof. As a computation shows, the equations of ζ_N in the coordinate systems introduced in the section 2.1.2, are as follows:

$$\begin{aligned} x^i \circ \zeta_N &= x^i, \\ y_{ij} \circ \zeta_N &= y_{ij}, \\ A_{ij}^h \circ \zeta_N &= \tfrac{1}{2} y^{hk}(y_{ik,j} + y_{jk,i} - y_{ij,k}), \quad i \leq j, \end{aligned} \tag{36}$$

where $(y^{ij})_{i,j=1}^n$ is the inverse mapping of the matrix $(y_{ij})_{i,j=1}^n$ and the functions y_{ij} are defined in (4). Hence

$$\begin{aligned} x^i \circ \zeta_N^{-1} &= x^i, \\ y_{ij} \circ \zeta_N^{-1} &= y_{ij}, \\ y_{ij,k} \circ \zeta_N^{-1} &= y_{hi} A_{jk}^h + y_{hj} A_{ik}^h, \quad i \leq j. \end{aligned} \tag{37}$$

As the diffeomorphism ζ_N induces the identity on the ground manifold N, it follows that the definition of γ^2 in (35) makes sense and the following formulas are obtained:

$$\gamma^2\left(\frac{\partial}{\partial x^r}\right)=\sum_{a\leq b}\left(\gamma_{abr}\circ\zeta_N\right)\frac{\partial}{\partial y_{ab}}+\sum_{i\leq j}\gamma_{ijkr}\frac{\partial}{\partial y_{ij,k}},$$

$$\begin{aligned}\gamma_{ijkr}&=\tfrac{1}{2}\sum_{a\leq b}\tfrac{\delta_{ah}\delta_{bi}+\delta_{ai}\delta_{bh}}{1+\delta_{hi}}\left(\gamma_{abr}\circ\zeta_N\right)y^{hl}(y_{jl,k}+y_{kl,j}-y_{jk,l})\\&+\tfrac{1}{2}\sum_{a\leq b}\tfrac{\delta_{ah}\delta_{bj}+\delta_{aj}\delta_{bh}}{1+\delta_{hj}}\left(\gamma_{abr}\circ\zeta_N\right)y^{hl}(y_{il,k}+y_{kl,i}-y_{ik,l})\\&+\sum_{j\leq a}\tfrac{\delta_{ak}}{1+\delta_{jk}}\left(\gamma^h_{jar}\circ\zeta_N\right)y_{hi}+\sum_{a\leq j}\tfrac{\delta_{ak}}{1+\delta_{jk}}\left(\gamma^h_{ajr}\circ\zeta_N\right)y_{hi}\\&+\sum_{i\leq a}\tfrac{\delta_{ak}}{1+\delta_{ik}}\left(\gamma^h_{iar}\circ\zeta_N\right)y_{hj}+\sum_{a\leq i}\tfrac{\delta_{ak}}{1+\delta_{ik}}\left(\gamma^h_{air}\circ\zeta_N\right)y_{hj},\end{aligned}$$

where

$$\gamma=\sum_{i\leq j}\left(dy_{ij}+\gamma_{ijk}dx^k\right)\otimes\frac{\partial}{\partial y_{ij}}+\sum_{j\leq k}\left(dA^i_{jk}+\gamma^i_{jkl}dx^l\right)\otimes\frac{\partial}{\partial A^i_{jk}},$$

or equivalently,

$$\gamma=\tfrac{1}{2-\delta_{ij}}\left(dy_{ij}+\gamma_{ijk}dx^k\right)\otimes\frac{\partial}{\partial y_{ij}}+\tfrac{1}{2-\delta_{jk}}\left(dA^i_{jk}+\gamma^i_{jkl}dx^l\right)\otimes\frac{\partial}{\partial A^i_{jk}},$$

assuming $\gamma_{hir}=\gamma_{ihr}$ for $h>i$, and $\gamma^h_{jkr}=\gamma^h_{kjr}$ for $j>k$. Taking the symmetry $A^i_{jk}=A^i_{kj}$ into account, we obtain

$$\begin{aligned}\gamma_{ijkr}&=\tfrac{1}{2}\left(\gamma_{hir}\circ\zeta_N\right)y^{hl}(y_{jl,k}+y_{kl,j}-y_{jk,l})\\&+\tfrac{1}{2}\left(\gamma_{hjr}\circ\zeta_N\right)y^{hl}(y_{il,k}+y_{kl,i}-y_{ik,l})\\&+\left(\gamma^h_{jkr}\circ\zeta_N\right)y_{hi}+\left(\gamma^h_{ikr}\circ\zeta_N\right)y_{hj}.\end{aligned}$$

Hence

$$\gamma_{ijkr}\circ\zeta_N^{-1}=\gamma_{hir}A^h_{jk}+\gamma_{hjr}A^h_{ik}+\gamma^h_{jkr}y_{hi}+\gamma^h_{ikr}y_{hj},\ i\leq j. \tag{38}$$

Permuting the indices i,j,k cyclically on the previous equation, we have

$$\gamma^s_{ijr}=-\gamma_{hkr}A^h_{ij}y^{ks}-\tfrac{1}{2}\left(\gamma_{ijkr}\circ\zeta_N^{-1}-\gamma_{jkir}\circ\zeta_N^{-1}-\gamma_{kijr}\circ\zeta_N^{-1}\right)y^{ks}, \tag{39}$$

thus proving that the mapping $\gamma\mapsto\gamma^2$ defined in the statement, is bijective. □

4.3 Covariant Hamiltonians for second-order Lagrangians

The Legendre form of a second-order Lagrangian density $\Lambda = Lv_n$ on the bundle $p\colon E \to N$ is the $V^*(p^1)$-valued p^3-horizontal $(n-1)$-form ω_Λ on J^3E locally given by (e.g., see [**17**], [**26**], [**35**]),

$$\omega_\Lambda = i_{\partial/\partial x^i} v_n \otimes \left(L_\alpha^{i0} dy^\alpha + L_\alpha^{ij} dy_j^\alpha \right),$$

where

$$L_\alpha^{ij} = \tfrac{1}{2-\delta_{ij}} \frac{\partial L}{\partial y_{ij}^\alpha}, \tag{40}$$

$$L_\alpha^{i} = \frac{\partial L}{\partial y_i^\alpha} - \sum_j \tfrac{1}{2-\delta_{ij}} D_j \left(\frac{\partial L}{\partial y_{ij}^\alpha} \right), \tag{41}$$

and

$$D_j = \frac{\partial}{\partial x^j} + \sum_{I\in\mathbb{N}^n, |I|=0}^{\infty} y_{I+(j)}^\alpha \frac{\partial}{\partial y_I^\alpha}$$

denotes the total derivative with respect to the variable x^j.

The Poincaré-Cartan form attached to Λ is then defined to be the ordinary n-form on J^3E given by, $\Theta_\Lambda = (p_2^3)^*\theta^2 \wedge \omega_\Lambda + \Lambda$, where θ^2 is the second-order structure form (cf. [**33**, (0.36)]) and the exterior product of $(p_2^3)^*\theta^2$ and the Legendre form, is taken with respect to the pairing induced by duality, $V(p^1) \times_{J^1E} V^*(p^1) \to \mathbb{R}$. The most outstanding difference with the first-order case is that the Legendre and Poincaré-Cartan forms associated with a second-order Lagrangian density are generally defined on J^3E, thus increasing by one the order of the density.

Similarly to the first-order case (see [**11**], [**24**]), given a second-order Lagrangian density Λ on $p\colon E \to N$ and a second-order connection γ^2 on $p\colon E \to N$, by subtracting $(p_2^3)^*\theta^2$ from $(p_1^3)^*\gamma^2$ we obtain a p^3-horizontal form, and we can define the corresponding covariant Hamiltonian to be the Lagrangian density Λ^{γ^2} of third order,

$$\Lambda^{\gamma^2} = \left((p_1^3)^*\gamma^2 - (p_2^3)^*\theta^2 \right) \wedge \omega_\Lambda - \Lambda. \tag{42}$$

Expanding on the right-hand side of the previous equation, we obtain a decomposition of Θ_Λ that generalizes the classical formula for the Hamiltonian in Mechanics; namely, $\Theta_\Lambda = (p_1^3)^*\gamma^2 \wedge \omega_\Lambda - \Lambda^{\gamma^2}$. With the same notations as in the formulas (34), (40), (41) the following formula is deduced:

$$L^{\gamma^2} = (\gamma_i^\alpha + y_i^\alpha) L_\alpha^{i0} + (\gamma_{hi}^\alpha + y_{hi}^\alpha) L_\alpha^{ih} - L. \tag{43}$$

Because of the equation (41), Θ_Λ and L^{γ^2} are generally defined on J^3E.

4.4 Invariant covariant Hamiltonians on J^2M

Lemma 4.2. *If γ is a first-order Ehresmann connection on $M \times_N C^{\mathrm{sym}}$ satisfying the conditions (C_M), then the following equation holds for the second-order Ehresmann connection γ^2 on M given in the formula* (35)*:*

$$\gamma_{abr} \circ \zeta_N = -y_{ab,r}.$$

Proof. Actually, from the formulas (23) and (36) we obtain

$$\begin{aligned}\gamma_{abr} \circ \zeta_N &= -\left(y_{mb}\left(A^m_{ra} \circ \zeta_N\right) + y_{ma}\left(A^m_{rb} \circ \zeta_N\right)\right) \\ &= -\tfrac{1}{2}\left\{y_{mb}y^{mk}(y_{rk,a} + y_{ak,r} - y_{ra,k}) + y_{ma}y^{mk}(y_{rk,b} + y_{bk,r} - y_{rb,k})\right\} \\ &= -y_{ab,r}.\end{aligned}$$

□

Lemma 4.3. *If a first-order connection γ on $M \times_N C^{\mathrm{sym}}$ satisfies the condition (C_C) introduced above, then the following formulas for its components hold:*

$$\gamma^h_{rts} - \gamma^h_{rst} = A^h_{sm}A^m_{rt} - A^h_{tm}A^m_{rs}. \tag{44}$$

Proof. As the bundle under consideration is that of symmetric connections, the following symmetry holds: $\gamma^h_{abc} = \gamma^h_{bac}$, and we have

$$\begin{aligned}\gamma^h_{rts} &= \gamma^h_{str} - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right) \quad \text{[by virtue of (28)]} \\ &= \gamma^h_{tsr} - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right) \\ &= \left(\gamma^h_{rst} + A^h_{rm}A^m_{st} - A^h_{tm}A^m_{rs}\right) \quad \text{[by virtue of (28)]} \\ &\quad - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right) \\ &= \gamma^h_{rst} + \left(A^h_{sm}A^m_{rt} - A^h_{tm}A^m_{rs}\right)\end{aligned}$$

□

Proposition 4.4. *Let*

$$\zeta^2_N = \zeta^{(1)}_N\Big|_{J^2M} : J^2M \to J^1(M \times_N C^{\mathrm{sym}})$$

be the restriction to the closed submanifold $J^2M \subset J^1(J^1M)$ of the prolongation $\zeta^{(1)}_N \colon J^1(J^1M) \to J^1(M \times_N C^{\mathrm{sym}})$ of the mapping ζ_N defined in Theorem 4.1. *For every $(j^1_x g, j^1_x\Gamma) \in J^1(M \times_N C^{\mathrm{sym}})$ there exists a unique $j^2_x g' \in J^2_x M$ such that, $j^1_x g' = j^1_x g$ and $j^1_x \Gamma^{g'} = j^1_x\Gamma$ and the mapping $\varkappa \colon J^1(M \times_N C^{\mathrm{sym}}) \to J^2M$ defined by $\varkappa(j^1_x g, j^1_x\Gamma) = j^2_x g'$ is a* $\mathrm{Diff}\,N$*-equivariant rectract of ζ^2_N.*

Proof. From the formulas (36) and (37) we obtain

$$\frac{\partial g'_{ij}}{\partial x^k} = g'_{hi}\left(\Gamma^{g'}\right)^h_{jk} + g'_{hj}\left(\Gamma^{g'}\right)^h_{ik},$$
$$\left(\Gamma^{g'}\right)^h_{ij} = \tfrac{1}{2}g'^{hk}\left(\frac{\partial g'_{ik}}{\partial x^j} + \frac{\partial g'_{jk}}{\partial x^i} - \frac{\partial g'_{ij}}{\partial x^k}\right)$$

for every non-singular metric g' on N. Hence the second partial derivatives of g'_{ij} are completely determined, namely

$$\frac{\partial^2 g'_{ij}}{\partial x^k \partial x^l} = \frac{\partial g_{hi}}{\partial x^l}\Gamma^h_{jk} + g_{hi}\frac{\partial \Gamma^h_{jk}}{\partial x^l} + \frac{\partial g_{hj}}{\partial x^l}\Gamma^h_{ik} + g_{hj}\frac{\partial \Gamma^h_{ik}}{\partial x^l}.$$

Moreover, the Levi-Civita connection of a metric depends functorially on the metric, i.e., $\phi \cdot \Gamma^g = \Gamma^{\phi \cdot g}$ for every $\phi \in \mathrm{Diff}\, N$. Hence, by transforming the equations $j^1_x g' = j^1_x g$ and $j^1_x \Gamma^{g'} = j^1_x \Gamma^g$ by ϕ we can conclude. □

Theorem 4.5. *If a first-order Ehresmann connection γ on $M \times_N C^{\mathrm{sym}}$ satisfies the conditions (C_M) and (C_C) introduced above, then the covariant Hamiltonian Λ^{γ^2} attached to every* $\mathrm{Diff}\, N$*-invariant second-order Lagrangian density Λ on M with respect to the second-order Ehresmann connection γ^2 on M defined in the formula* (35)*, is defined on J^2M and it is also* $\mathrm{Diff}\, N$*-invariant.*

Proof. Given a $\mathrm{Diff}\, N$-invariant second-order Lagrangian density $\Lambda = \mathcal{L}\mathbf{v}$ on M, let $\Lambda' = \mathcal{L}'\mathbf{v}$ be the first-order Lagrangian density on $M \times_N C^{\mathrm{sym}}$ given by $\Lambda' = \varkappa^*\Lambda$, which is also $\mathrm{Diff}\, N$-invariant as $\varkappa$ is a $\mathrm{Diff}\, N$-equivariant mapping according to Proposition 4.4. Moreover, as $\varkappa$ is a retract of ζ^2_N, we have $\left(\zeta^2_N\right)^* \Lambda' = \left(\zeta^2_N\right)^* \varkappa^*\Lambda = (\varkappa \circ \zeta^2_N)^*\Lambda = \Lambda$, i.e., $\Lambda = \left(\zeta^2_N\right)^* \Lambda'$. This formula is equivalent to saying $\mathcal{L} = \mathcal{L}' \circ \zeta^2_N$, as the n-form $\mathbf{v}$ is $\mathrm{Diff}\, N$-invariant, and it is even equivalent to $L = L' \circ \zeta^2_N$ because ζ^2_N induces the identity on N.

We claim $\mathcal{L}^{\gamma^2} = (\mathcal{L}')^{\gamma} \circ \zeta^2_N$. This formula will end the proof as the mapping ζ^2_N is $\mathrm{Diff}\, N$-equivariant and $(\mathcal{L}')^{\gamma}$ is $\mathrm{Diff}\, N$-invariant by virtue of Theorem 3.6.

To start with, we observe that the formula (40) for Λ can be written, in the present case, as follows:

$$L^{abij} = \tfrac{1}{2-\delta_{ij}}\frac{\partial L}{\partial y_{ab,ij}},$$

or equivalently, letting $\mathcal{L}^{abij} = \rho^{-1} L^{abij}$,

$$\mathcal{L}^{abij} = \tfrac{1}{2-\delta_{ij}} \frac{\partial \mathcal{L}}{\partial y_{ab,ij}}. \tag{45}$$

Taking the formula in Lemma 4.2 into account, the formula (43) for Λ reads as $L^{\gamma^2} = \sum_{a\leq b}(\gamma_{abij} + y_{ab,ij})L^{abij} - L$, or even

$$\mathcal{L}^{\gamma^2} = \sum_{a\leq b}(\gamma_{abij} + y_{ab,ij})\mathcal{L}^{abij} - \mathcal{L},$$

where $\mathcal{L}^{\gamma^2} = \rho^{-1} L^{\gamma^2}$. Hence $\mathcal{L}^{\gamma^2}$ is defined over J^2M. As $y_{ab,ij} = y_{ab,ji}$, we obtain

$$\begin{aligned}
\mathcal{L}^{\gamma^2} &= \sum_{a\leq b}\sum_{i\leq j}\left(\tfrac{1}{2}\left(\gamma_{abij} + \gamma_{abji}\right) + y_{ab,ij}\right) \frac{\partial\left(\mathcal{L}' \circ \zeta_N^2\right)}{\partial y_{ab,ij}} - \mathcal{L}' \circ \zeta_N^2 \\
&= \sum_{a\leq b}\sum_{i\leq j}\sum_{k\leq l}\left(\tfrac{1}{2}\left(\gamma_{abij} + \gamma_{abji}\right) + y_{ab,ij}\right)\left(\frac{\partial \mathcal{L}'}{\partial A^h_{kl,q}} \circ \zeta_N^2\right) \frac{\partial (A^h_{kl,q} \circ \zeta_N^2)}{\partial y_{ab,ij}} - \mathcal{L}' \circ \zeta_N^2 \\
&= \sum_{k\leq l} \tfrac{1}{4} y^{hm}\left(\gamma_{kmql} + \gamma_{kmlq} + \gamma_{lmqk} + \gamma_{lmkq} - \gamma_{klqm} - \gamma_{klmq}\right)\left(\frac{\partial \mathcal{L}'}{\partial A^h_{kl,q}} \circ \zeta_N^2\right) \\
&\quad + \sum_{k\leq l} \tfrac{1}{2} y^{hm}\left(y_{km,ql} + y_{lm,qk} - y_{kl,qm}\right)\left(\frac{\partial \mathcal{L}'}{\partial A^h_{kl,q}} \circ \zeta_N^2\right) - \mathcal{L}' \circ \zeta_N^2.
\end{aligned}$$

Moreover, we have

$$\left(\mathcal{L}'\right)^{\gamma} = \sum_{a\leq b}\left(\gamma_{abc} + y_{ab,c}\right)\frac{\partial \mathcal{L}'}{\partial y_{ab,c}} + \sum_{a\leq b}\left(\gamma^i_{abl} + A^i_{ab,l}\right)\frac{\partial \mathcal{L}'}{\partial A^i_{ab,l}} - \mathcal{L}'.$$

Hence

$$\begin{aligned}
\left(\mathcal{L}'\right)^{\gamma} \circ \zeta_N^2 &= \sum_{k\leq l}\left(\gamma^h_{klq} \circ \zeta_N + A^h_{kl,q} \circ \zeta_N\right)\left(\frac{\partial \mathcal{L}'}{\partial A^h_{kl,q}} \circ \zeta_N^2\right) - \mathcal{L}' \circ \zeta_N^2 \\
&= \sum_{k\leq l}\Big\{-\tfrac{1}{2}\left(\gamma_{klrq} - \gamma_{lrkq} - \gamma_{rklq}\right) y^{rh} \\
&\quad + \tfrac{1}{2}\left(y_{kr,lq} + y_{lr,kq} - y_{kl,rq}\right) y^{hr}\Big\}\left(\frac{\partial \mathcal{L}'}{\partial A^h_{kl,q}} \circ \zeta_N^2\right) - \mathcal{L}' \circ \zeta_N^2.
\end{aligned}$$

Consequently, the proof reduces to state that the following equation

$$\tfrac{1}{4}\left(\gamma_{krql} + \gamma_{krlq} + \gamma_{lrqk} + \gamma_{lrkq} - \gamma_{klqr} - \gamma_{klrq}\right) = -\tfrac{1}{2}\left(\gamma_{klrq} - \gamma_{lrkq} - \gamma_{rklq}\right)$$

holds true, or equivalently,

$$0 = \left(\gamma_{ijkr} - \gamma_{ijrk}\right) + \left(\gamma_{irjk} - \gamma_{irkj}\right) + \left(\gamma_{rjki} - \gamma_{rjik}\right). \tag{46}$$

According to the formulas (38) and (23) we obtain

$$\begin{aligned}\gamma_{ijkr}\circ\zeta_N^{-1} &= \left(\gamma^h_{jkr}-A^h_{ra}A^a_{jk}\right)y_{hi}+\left(\gamma^h_{ikr}-A^h_{ra}A^a_{ik}\right)y_{hj}\\ &\quad-\left(A^h_{rj}A^a_{ik}+A^h_{ri}A^a_{jk}\right)y_{ah}.\end{aligned}$$

The third term on the right-hand side of this equation is symmetric in the indices k and r, as $A^a_{bc}=A^a_{cb}$. Hence

$$\begin{aligned}(\gamma_{ijkr}-\gamma_{ijrk})\circ\zeta_N^{-1} &= \left(\gamma^h_{jkr}-\gamma^h_{jrk}-A^h_{ra}A^a_{jk}+A^h_{ka}A^a_{jr}\right)y_{hi}\\ &\quad+\left(\gamma^h_{ikr}-\gamma^h_{irk}-A^h_{ra}A^a_{ik}+A^h_{ka}A^a_{ir}\right)y_{hj}.\end{aligned}$$

By composing the right-hand side of the equation (46) and ζ_N^{-1}, and taking the previous formula and the formulas (28) and (44) into account, we conclude that this expression vanishes indeed. □

5 Palatini and Einstein-Hilbert Lagrangians

Let us compute the covariant Hamiltonian density attached to the Palatini Lagrangian. Following the notations in [**20**], the Ricci tensor field attached to the symmetric connection Γ is given by $S^\Gamma(X,Y)=\mathrm{tr}(Z\mapsto R^\Gamma(Z,X)Y)$, where R^Γ denotes the curvature tensor field of the covariant derivative ∇^Γ associated to Γ on the tangent bundle; hence $S^\Gamma=(R^\Gamma)_{jl}dx^l\otimes dx^j$, where

$$\begin{aligned}(R^\Gamma)_{jl} &= (R^\Gamma)^k_{jkl},\\ (R^\Gamma)^i_{jkl} &= \partial\Gamma^i_{jl}/\partial x^k-\partial\Gamma^i_{jk}/\partial x^l+\Gamma^m_{jl}\Gamma^i_{km}-\Gamma^m_{jk}\Gamma^i_{lm}.\end{aligned}$$

The Lagrangian is the function on $J^1(M\times_N C^{\mathrm{sym}})$ thus given by,

$$\mathcal{L}_P(j^1_x g, j^1_x\Gamma)=g^{ij}(x)(R^\Gamma)_{ij}(x)$$

and local expression

$$\mathcal{L}_P=y^{ij}(A^k_{ij,k}-A^k_{ik,j}+A^m_{ij}A^k_{km}-A^m_{ik}A^k_{jm}).$$

As a computation shows, for every first-order connection γ on $M\times_N C^{\mathrm{sym}}$ satisfying (44) and taking the formula (2) into account, we obtain $\mathcal{L}_P^\gamma=0$. This result is essentially due to the fact that the P-C form of the P density $\Lambda_P=\mathcal{L}_P\mathbf{v}=L_P v_n$ projects onto $M\times_N C^{\mathrm{sym}}$. In fact, the following general characterization holds:

Proposition 5.1. *Let $p\colon E\to N$ be an arbitrary fibred manifold and let γ be a first-order Ehresmann connection on E. The equation $L^\gamma=0$ holds true for a Lagrangian $L\in C^\infty(J^1E)$ if and only if,* i) *the Poincaré-Cartan form of the density $\Lambda=Lv_n$ projects onto J^0E and,* ii) $L=\left\langle (p^1_0)^*\gamma-\theta, dL|_{V(p^1_0)}\right\rangle$.

Proof. The equation $L^\gamma = 0$ is equivalent to the equation $D^\gamma L = L$, where D^γ is the p_0^1-vertical vector field defined in the formula (17), and the general solution to the latter is $L = f(x^i, y^\alpha, \gamma_i^\alpha + y_i^\alpha)$, $f(x^i, y^\alpha, y_i^\alpha)$ being a homogeneous smooth function of degree one in the variables (y_i^α), $1 \leq \alpha \leq m$, $1 \leq i \leq n$, according to Euler's homogeneous function theorem. As f is defined for all values of the variables (y_i^α), $1 \leq \alpha \leq m$, $1 \leq i \leq n$, we conclude that the functions $L_\alpha^i = \partial L/\partial y_i^\alpha$ must be defined on E. Hence L is written as $L = L_\alpha^i(x^j, y^\beta) y_i^\alpha + L_0(x^j, y^\beta)$, but this is exactly the condition for the P-C form of Λ to be projectable onto $J^0E = E$, as follows from the local expression of this form, namely,

$$
\begin{aligned}
\Theta_\Lambda &= \frac{\partial L}{\partial y_i^\alpha} \theta^\alpha \wedge i_{\partial/\partial x^i} v_n + L v_n \\
&= \frac{\partial L}{\partial y_i^\alpha} dy^\alpha \wedge i_{\partial/\partial x^i} v_n + \left(L - y_i^\alpha \frac{\partial L}{\partial y_i^\alpha} \right) v_n.
\end{aligned}
$$

Moreover, by imposing the condition $D^\gamma L = L$ we obtain $L_0 = L_\alpha^i \gamma_i^\alpha$, or in other words $L = (\gamma_i^\alpha + y_i^\alpha) \partial L/\partial y_i^\alpha$, which is equivalent to the equation ii) in the statement. □

The corresponding result for the second-order formalism is similar but the computations are more cumbersome. Let us compute the covariant Hamiltonian density attached to the Einstein-Hilbert Lagrangian. As a matter of notation, we set $S^g(X,Y) = S^{\Gamma^g}(X,Y)$ for the metric g, Γ^g being its Levi-Civita connection, and similarly, $(R^g)^i_{jkl} = (R^{\Gamma^g})^i_{jkl}$.

The E-H Lagrangian is thus given by $\mathcal{L}_{EH} \circ j^2 g = (y^{ij} \circ g)(R^g)^h_{ihj}$. As the Levi-Civita connection Γ^g depends functorially on g, $\mathcal{L}_{EH}$ is readily seen to be Diff$N$-invariant; it is in addition linear in the second-order variables $y_{ij,kl}$. By using the third formula in (36) the following local expression for $\mathcal{L}_{EH}$ is obtained:

$$
\mathcal{L}_{EH} = \tfrac{1}{2} y^{ij} y^{hd} \left(y_{dj,hi} - y_{ij,dh} - y_{dh,ij} + y_{hi,dj} \right) + \mathcal{L}'_{EH},
$$

$$
\begin{aligned}
\mathcal{L}'_{EH} = \tfrac{1}{2} y^{ij} \Big\{ & y^{hm} y_{mr,j} y^{rd} \left(y_{id,h} + y_{hd,i} - y_{ih,d} \right) \\
& - y^{hm} y_{mr,h} y^{rd} \left(y_{id,j} + y_{jd,i} - y_{ij,d} \right) \\
& + \tfrac{1}{2} y^{hr} y^{md} \left(y_{id,j} + y_{jd,i} - y_{ij,d} \right) \left(y_{hr,m} + y_{mr,h} - y_{hm,r} \right) \\
& - \tfrac{1}{2} y^{hr} y^{md} \left(y_{id,h} + y_{hd,i} - y_{ih,d} \right) \left(y_{jr,m} + y_{mr,j} - y_{jm,r} \right) \Big\}.
\end{aligned}
$$

According to (45), for every first-order connection form γ on $M \times_N C^{\mathrm{sym}}$ satisfying the conditions (C_M) and (C_C) above, we have

$$\mathcal{L}_{EH}^{\gamma^2} = \sum_{a \le b} \tfrac{1}{2-\delta_{ij}} (\gamma_{abij} + y_{ab,ij}) \frac{\partial \mathcal{L}_{EH}}{\partial y_{ab,ij}} - \mathcal{L}_{EH},$$

and as a computation shows,

$$\begin{aligned}
\mathcal{L}_{EH}^{\gamma^2} &= \tfrac{1}{2} y^{ij} \left(\gamma_{idjh} + \gamma_{jdih} - \gamma_{ijdh} - \gamma_{idhj} - \gamma_{hdij} + \gamma_{ihdj}\right) y^{hd} \\
&+ \tfrac{1}{2} y^{ij} \Big\{ y^{hm} y_{mr,h} y^{rd} \left(y_{id,j} + y_{jd,i} - y_{ij,d}\right) \\
&- y^{hm} y_{mr,j} y^{rd} \left(y_{id,h} + y_{hd,i} - y_{ih,d}\right) \\
&- \tfrac{1}{2} y^{hr} y^{md} \left(y_{id,j} + y_{jd,i} - y_{ij,d}\right)\left(y_{hr,m} + y_{mr,h} - y_{hm,r}\right) \\
&+ \tfrac{1}{2} y^{hr} y^{md} \left(y_{id,h} + y_{hd,i} - y_{ih,d}\right)\left(y_{jr,m} + y_{mr,j} - y_{jm,r}\right) \Big\} \\
&= 0,
\end{aligned}$$

where the formulas (39), (44), (36), and Lemma 4.3 have been used. In this case, the P-C form of the E-H density $\Lambda_{EH} = \mathcal{L}_{EH}\mathbf{v} = L_{EH} v_n$,

$$\begin{aligned}
(47) \qquad \Theta_{\Lambda_{EH}} &= \sum\nolimits_{k \le l} \left(L_{EH}^{i,kl} dy_{kl} + L_{EH}^{ij,kl} dy_{kl,j} \right) \wedge i_{\partial/\partial x^i} v_n + H v_n, \\
H &= L'_{EH} - \sum\nolimits_{k \le l} L_{EH}^{i,kl} y_{kl,i}, \\
L_{EH}^{i,kl} &= \frac{\partial L'_{EH}}{\partial y_{kl,i}} - \tfrac{1}{2-\delta_{ij}} y_{ab,j} \frac{\partial^2 L_{EH}}{\partial y_{ab} \partial y_{kl,ij}}, \\
L_{EH}^{ij,kl} &= \tfrac{1}{2-\delta_{ij}} \frac{\partial L_{EH}}{\partial y_{kl,ij}},
\end{aligned}$$

(cf. (40), (41)) is not only projectable onto $J^2 M$ but also on $J^1 M$ (e.g., see [**13**]), although there is no first-order Lagrangian on $J^1 M$ admitting (47) as its P-C form. This fact is strongly related to a classical result by Hermann Weyl ([**39**, Appendix II], also see [**22**], [**18**]) according to which the only $\mathrm{Diff}\,N$-invariant Lagrangians on $J^2 M$ depending linearly on the second-order coordinates $y_{ab,ij}$ are of the form $\lambda \mathcal{L}_{EH} + \mu$, for scalars λ, μ. This also explains why a true first-order Hamiltonian formalism exists in the Einstein-Cartan gravitation theory, e.g., see [**37**], [**38**]. In fact, if

$$L_{EH}^{i} = \tfrac{1}{2-\delta_{ij}} \frac{\partial L_{EH}}{\partial y_{kl,ij}} y_{kl,j} \qquad \left(\text{hence } \; L_{EH}^{ij,kl} = \frac{\partial L_{EH}^{i}}{\partial y_{kl,j}} \right)$$

and the momentum functions are defined as follows:

$$p_{kl,i} = L_{EH}^{i,kl} - \frac{\partial L_{EH}^{i}}{\partial y_{kl}},$$

then

$$d\Theta_{\Lambda_{EH}} = dp_{kl,i} \wedge dy_{kl} \wedge i_{\partial/\partial x^i} v_n + dH \wedge v_n,$$

and from the Hamilton-Cartan equation (e.g., see [**13**, (1)]) we conclude that a metric g is an extremal for Λ_{EH} if and only if,

$$0 = \frac{\partial(p_{ab,i} \circ j^1 g)}{\partial x^i} - \frac{\partial H}{\partial y_{ab}} \circ j^1 g,$$
$$0 = \frac{\partial(y_{ab} \circ g)}{\partial x^i} + \frac{\partial H}{\partial y_{ab,i}} \circ j^1 g.$$

On the other hand, it is no longer true that the covariant Hamiltonians of the non-linear Lagrangians of the form $f(\mathcal{L}_{EH})$, $f'' \neq 0$, considered in some cosmological models (e.g., see [**1**], [**6**], [**9**], [**12**], [**19**], [**21**], [**31**]) and those in higher dimensions (e.g., see [**15**], [**36**]) vanish. In fact, as a computation shows, one has $f(\mathcal{L}_{EH})^{\gamma^2} = f'(\mathcal{L}_{EH})\mathcal{L}_{EH} - f(\mathcal{L}_{EH})$, $\forall f \in C^\infty(\mathbb{R})$.

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